Relaxation of the nematic deformation when the distorting field is removed

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We investigate the relaxation of the nematic deformation when the distorting field is switched off. We show that the usual analysis based on the diffusionlike equation does not allow a complete description of the phenomenon because it does not permit one to satisfy the initial boundary condition, at $t=0$, on the first time derivative of the nematic tilt angle. An alternative approach to the problem, taking into account the inertial properties of the nematic molecules, allows one to satisfy the initial boundary conditions on the first-order time derivative of the tilt angle. In this framework the dynamical evolution of the nematic deformation, in the initial time, depends on the inertial properties of the nematic molecules. However, the typical relaxation time is so short that, for all practical effects, the first time derivative of the tilt angle is discontinuous at $t=0$. A more realistic description involves the switching time of the distorting field. In this framework, the initial boundary condition of the first-order derivative is automatically satisfied. Our analysis shows that the description based on the diffusion equation works well when the switching time is very small with respect to the diffusion time.

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I. INTRODUCTION

Nematic liquid crystals are anisotropic liquid. Due to their anisotropy they interact with electric and magnetic fields. The nematic liquid crystal displays are based on the interaction of the nematic dielectric anisotropy with an external electric field. We are interested in the relaxation of an imposed deformation when the distorting field is removed. This problem has been analyzed by several groups [1–9]. The standard theoretical analysis of this phenomenon is based on the diffusion equation, where the elastic torque is balanced by the viscous torque [10,11]. In this description, the diffusion equation is solved with the initial boundary condition that initial nematic profile coincides with the deformation in the presence of the distorting field. According to this model, the first time derivative of the nematic distortion is discontinuous at the instant when the field is removed. To solve this inconsistence of the model, it has been proposed to take into account the inertial moment of the nematic molecules, in such a manner that the dynamical equation of the problem is of second order in time. However, the typical relaxation time related to the inertial properties of the nematic molecules is so short that, for all practical effects, the first time derivative of the tilt angle is discontinuous at $t=0$. We show that to describe the phenomenon in a proper manner it is necessary to take into account that the switching time of the distorting field is finite. In this framework, the continuity of the first time derivative of the nematic deformation is automatically satisfied. Our paper is organized as follows. The system under investigation is described in Sec. II. The standard analysis based on the diffusion equation is presented in Sec. III. The influence of the inertial properties of the nematic molecules on the relaxation phenomenon is discussed in Sec. IV. The role of the switching time on the relaxation of the imposed deformation is investigated in Sec. V. In Sec. VI we compare the predictions of the considered models, and the final section, Sec. VII, is devoted to the conclusions. The Appendix is devoted to the derivation of the electrostatic energy density of a nematic liquid crystal subjected to an external field, in the limit of small deformation of the director field.

II. POSITION OF THE PROBLEM

We discuss the relaxation of a deformation induced by an external field in a nematic liquid crystal cell when the distorting field is removed. The nematic orientation is defined by a vectorial field **n**, coinciding with the statistical average of the molecular direction [12]. We consider a nematic cell in the shape of a slab of thickness d. The Cartesian reference frame used in the description has the z axis normal to the limiting surfaces, located at $z=\pm d/2$. The surfaces of the slab are treated in such a manner to have the easy axes parallel to each other, at an angle ϕ_s with the z axis [13]. In this framework the nematic deformation is contained in a plane we indicate by (x,z). In the following the surface anchoring energy is supposed to be strong, in such a manner that the surface nematic orientation coincides with the easy direction for all imposed deformation. The nematic liquid crystal is assumed to have a positive dielectric anisotropy [12], and the distorting electric field, **E**, parallel to the z axis. In this situation, the electric field tends to align the nematic director along its direction.

Let us consider now the nematic sample submitted to a constant electric field $\mathbf{E_0}=E_0\mathbf{z}$ (see Appendix). The actual nematic orientation, $\Phi=\Phi(z)$, is the one where the bulk density of elastic torque is balanced by the bulk density of electric torque [14]

$$k\frac{d^2\Phi}{dz^2}-\frac{1}{2}\varepsilon_a E_0^2\sin(2\Phi)=0, \tag{1}$$

where k is the elastic constant of Frank, Φ is the angle formed by the nematic director with the z axis, and ε_a the dielectric anisotropy of the liquid crystal. Equation (1) is valid in the one-constant approximation, over which will be based our analysis [12]. The stable $\Phi=\Phi(z)$ is the solution of Eq. (1) that satisfies the boundary conditions

$$\Phi(\pm d/2)=\phi_s, \tag{2}$$

related to the strong anchoring hypothesis. In our analysis we assume that ϕ_s is small, in such a manner that $\sin(2\phi_s)$

$\sim 2\phi_s$. Since $\Phi(z) < \phi_s$, Eq. (1) can be linearized, and the fundamental equation of the problem reads

$$k\frac{d^2\Phi}{dz^2} - \varepsilon_a E_0^2 \Phi = 0. \tag{3}$$

The solution of Eq. (3) with the boundary conditions (2) is

$$\Phi(z) = \phi_s \frac{\cosh(z/\lambda)}{\cosh(d/2\lambda)}, \tag{4}$$

where $\lambda = (1/E_0)\sqrt{k/\varepsilon_a}$ is the electric coherence length [12]. In the following the solution given by Eq. (4) will be written as

$$\Phi(z) = \phi_s + \delta\varphi(z), \quad \text{where} \quad \delta\varphi(z) = \phi_s\left(\frac{\cosh(z/\lambda)}{\cosh(d/2\lambda)} - 1\right). \tag{5}$$

For reasons that will be clear in the following, it is useful to decompose $\delta\varphi(z)$ is series of Fourier as follows:

$$\delta\varphi(z) = \sum_{n=0}^{\infty} D_n \cos(a_n z), \tag{6}$$

where

$$a_n = (2n+1)(\pi/d), \quad \text{and} \quad D_n = -4\phi_s \frac{(-1)^n}{a_n[1+(a_n\lambda)^2]}. \tag{7}$$

As it is clear from the discussion reported above, when the electric field is constant, the tilt angle depends just on z, and its time derivative is identically zero. We are interested now in the relaxation of the deformation $\Phi(z)$ when the distorting electric field is switched off. We assume that the electric field is removed at $t=0$. For $t>0$, the nematic tilt angle depends on z and t, $\phi=\phi(z,t)$, and it is such that

$$\lim_{t\to 0} \phi(z,t) = \Phi(z), \quad \text{and} \quad \lim_{t\to\infty} \phi(z,t) = \phi_s. \tag{8}$$

In the following sections we describe the different models proposed to describe the evolution of the tilt angle we are looking for.

III. STANDARD ANALYSIS

In the standard analysis [10] the relaxation of the initial deformation is described by the partial differential equation

$$k\frac{\partial^2\phi_D}{\partial z^2} = \eta\frac{\partial\phi_D}{\partial t}, \tag{9}$$

stating that during the relaxation, the elastic torque is balanced by the viscous torque. In Eq. (9) the coefficient η is the rotational viscosity of the nematic liquid crystal [12]. Equation (9) holds true when the inertial moment of the nematic molecules is negligible, and the electric field is removed suddenly, with switching time zero. It is written by neglecting the hydrodynamic backflow induced by the reorientation of the nematic director [15]. This approximation works well only in the case where the deformation of the nematic liquid crystal is small, as we will suppose in the following. We look for a solution of Eq. (9) of the type

$$\phi_D(z,t) = \phi_s + \delta\psi(z,t), \tag{10}$$

where, as it follows from Eqs. (8),

$$\delta\psi(z,0) = \delta\varphi(z), \quad \text{and} \quad \lim_{t\to\infty} \delta\psi(z,t) = 0. \tag{11}$$

Since the anchoring is strong, we have also the boundary condition

$$\delta\psi(\pm d/2,t) = 0. \tag{12}$$

By taking into account Eq. (12), the solution of Eq. (9), with the boundary conditions (11), is

$$\delta\psi(z,t) = \sum_{n=0}^{\infty} D_n \cos(a_n z)\exp\left(-\frac{k}{\eta}a_n^2 t\right). \tag{13}$$

The relaxation times are given by

$$\tau_n = \frac{\eta}{k}\frac{1}{a_n^2} = \frac{\tau_D}{(2n+1)^2\pi^2}, \tag{14}$$

where $\tau_D = \eta d^2/k$ is the diffusion time. The analysis presented above looks coherent. However, there are a few black points. For $t<0$ the actual nematic profile is such that its time derivative is identically zero. On the contrary, according to Eq. (9), for $t>0$ we have

$$\frac{\partial\phi_D}{\partial t} = \frac{k}{\eta}\frac{\partial^2\phi_D}{\partial z^2}. \tag{15}$$

In particular, from Eq. (15), taking into account Eq. (13), we get

$$\left(\frac{\partial\phi_D}{\partial t}\right)_{t=0^+} = \frac{k}{\eta}\frac{d^2\Phi}{dz^2}, \tag{16}$$

that for Eq. (4) can be rewritten as

$$\left(\frac{\partial\phi_D}{\partial t}\right)_{t=0^+} = \frac{k}{\eta\lambda^2}\phi_s\frac{\cosh(z/\lambda)}{\cosh(d/2\lambda)}. \tag{17}$$

From Eq. (16) it follows that the first-order time derivative of the tilt angle is discontinuous at $t=0$, and hence the second-order time derivative has, at $t=0$, a δ-Dirac behavior. Equation (17) indicates that the typical time over which $\partial\phi/\partial t$ is varying, at $t=0$, is of the order of the relaxation time connected to the coherence length λ, $\tau_\lambda = \eta\lambda^2/k \ll \tau_D$.

From Eq. (15) we obtain that the second time derivative of the tilt angle is given by

$$\frac{\partial^2\phi_D}{\partial t^2} = \frac{k}{\eta}\frac{\partial}{\partial t}\left(\frac{\partial^2\phi_D}{\partial z^2}\right) = \frac{k}{\eta}\frac{\partial^2}{\partial z^2}\left(\frac{\partial\phi_D}{\partial t}\right) = \left(\frac{k}{\eta}\right)^2\frac{\partial^4\phi_D}{\partial z^4}. \tag{18}$$

For $t=0$ the initial rotational acceleration is then

$$\left(\frac{\partial^2\phi_D}{\partial t^2}\right)_{t=0^+} = \left(\frac{k}{\eta}\right)^2\frac{d^4\Phi}{dz^4} = \left(\frac{k}{\eta\lambda^2}\right)^2\phi_s\frac{\cosh(z/\lambda)}{\cosh(d/2\lambda)}, \tag{19}$$

indicating that the typical time is τ_λ.

According to the analysis presented above, it is impossible, in the framework of the model based on the diffusion equation, Eq. (9), to satisfy the initial condition on the first time derivative. The second time derivative is also discontinuous at $t=0$, and exhibits a divergence, of the type of δ-Dirac at $t=0$.

IV. INERTIAL CONTRIBUTION TO THE TOTAL TORQUE EQUILIBRIUM

A possible way to satisfy the initial condition on the first-order time derivative of the nematic director is to take into account the inertial contribution to the total torque. According to elementary mechanics [16], the dynamical equation for the nematic director, when the inertial contribution is not negligible, is

$$k\frac{\partial^2\phi_I}{\partial z^2}=\eta\frac{\partial\phi_I}{\partial t}+I\frac{\partial^2\phi_I}{\partial t^2}, \tag{20}$$

where the I is the molecular inertial momentum, per unit volume. Equation (20) has to be solved with the time boundary conditions

$$\phi_I(z,0)=\Phi(z), \quad \text{and} \quad \left(\frac{\partial\phi_I}{\partial t}\right)_{t=0}=0, \tag{21}$$

related to the continuity of the function and of its time derivative, at $t=0$,

$$\lim_{t\to\infty}\phi_I(z,t)=\phi_s, \quad \text{and} \quad \lim_{t\to\infty}\frac{\partial\phi_I}{\partial t}=0, \tag{22}$$

related to the stable state imposed by the surface treatment, and $\phi(\pm d/2,t)=\phi_s$, due to the strong anchoring hypothesis.

We look for a solution of Eq. (20), with the boundary conditions Eqs. (21) and (22), of the type

$$\phi_I(z,t)=\phi_s+\sum_{n=0}^{\infty}C_n(t)\cos(a_n z), \tag{23}$$

where, as before, $a_n=(2n+1)(\pi/d)$, and the functions $C_n(t)$ are such that

$$C_n(0)=D_n, \quad \left(\frac{dC_n}{dt}\right)_{t=0}=0, \quad \text{and} \quad \lim_{t\to\infty}C_n(t)=0. \tag{24}$$

By substituting the ansatz (23) into Eq. (20), taking into account the linear independence of the set of functions $\cos(a_n z)$ we get that the coefficients $C_n(t)$ are solutions of the differential equations

$$\frac{I}{k}\frac{d^2C_n}{dt^2}+\frac{\eta}{k}\frac{dC_n}{dt}+a_n^2C_n=0. \tag{25}$$

The solutions of Eq. (25), satisfying the conditions (24) are

$$C_n(t)=T_{n1}\exp(-\beta_{n1}t)+T_{n2}\exp(-\beta_{n2}t), \tag{26}$$

where

$$\beta_{n1}=\frac{\eta}{2I}+\sqrt{\left(\frac{\eta}{2I}\right)^2-\frac{k}{I}a_n^2},$$

$$\beta_{n2}=\frac{\eta}{2I}-\sqrt{\left(\frac{\eta}{2I}\right)^2-\frac{k}{I}a_n^2}, \tag{27}$$

and

$$T_{n1}=\frac{\beta_{n2}}{\beta_{n2}-\beta_{n1}}D_n,$$

$$T_{n2}=-\frac{\beta_{n1}}{\beta_{n2}-\beta_{n1}}D_n. \tag{28}$$

In the limit of $I\to 0$ from Eqs. (27) we get

$$\beta_{n1}\sim\frac{\eta}{I}, \quad \text{and} \quad \beta_{n2}\sim\frac{k}{\eta}a_n^2. \tag{29}$$

The relaxation times are $\tau_{n1}=1/\beta_{n1}$ and $\tau_{n2}=1/\beta_{n2}$. For $I\to 0$, corresponding to the previous case, the characteristic exponents tend to $\beta_{n1}\to\infty$, and $\beta_{n2}\to ka_n^2/\eta$, and hence $\tau_{n1}\to 0$ and $\tau_{n2}\to\eta/(ka_n^2)$, whereas the coefficients, in the same limit, tend to $T_{n1}\to 0$, and $T_{n2}=D_n$, as expected [see Eq. (13)].

In the model under consideration, the initial angular acceleration of the nematic director, as follows from Eq. (20), taking into account the boundary conditions (21), is

$$\left(\frac{\partial^2\phi_I}{\partial t^2}\right)_{t=0}=\frac{k}{I}\frac{d^2\Phi}{dz^2}=\frac{k}{I\lambda^2}\phi_s\frac{\cosh(z/\lambda)}{\cosh(d/2\lambda)}. \tag{30}$$

As already observed in the previous section, when $I=0$, the initial angular acceleration of the nematic director diverges.

To have an idea about the importance of the inertial contribution to the dynamics of the relaxation, we write Eq. (20) in terms of dimensionless coordinates $z_r=z/d$ and $t_r=t/\tau_D$, where $\tau_D=(\eta/k)d^2$ is the diffusion time introduced above. We get

$$\frac{\partial^2\phi_I}{\partial z_r^2}=\frac{\partial\phi_I}{\partial t_r}+\nu\frac{\partial^2\phi}{\partial t_r^2}, \tag{31}$$

where

$$\nu=\frac{kI}{\eta^2d^2}, \tag{32}$$

is the dimensionless small parameter taking into account the inertial properties of the nematic molecules. The bulk density of moment of inertia of the nematic molecules is defined by $I=\rho b^2$, where ρ is the bulk density and b a molecular dimension. For typical nematic liquid crystals I is of the order of 10^{-15} kg/m [17]. By assuming $k=10^{-11}$ kg/m, $\eta=10^{-1}$ Pa s, $d=10^{-6}$ m [17] we obtain $\nu\sim 10^{-12}$. This number is extremely small. Consequently, for actual values of the nematic parameters, the importance of the inertial properties is limited to the initial time. For times of the order of τ_D, the role of the inertial contribution to the equilibrium of the torques in the bulk is negligible. For this reason, in the following we will consider very large values for the density of the momentum of inertia (25×10^{-12} kg/m$\leq I\leq 25\times 10^{-8}$ kg/m), just to show the effect of the inertial term in the evolution of the nematic deformation when the field is switched off.

V. INFLUENCE OF THE SWITCHING TIME OF THE DISTORTING FIELD ON THE RELAXATION PROCESS

In the previous sections, the distorting field was assumed to be removed in a discontinuous manner. This means that the switching time was supposed zero. Of course, in real system, the switching time is finite. In this section we analyze the influence of a finite switching time on the relaxation of the initial deformation of the nematic liquid crystal. We will base our investigation of the equation

$$k\frac{\partial^2\phi_E}{\partial z^2}-\varepsilon_a E^2(t)\phi_E=\eta\frac{\partial\phi_E}{\partial t}, \tag{33}$$

where $E^2(t)=E_0^2 f(t)$, with $f(t)=1$ for $t\leq 0$, and $f(t)\to 0$ for $t\to\infty$. In the following we will consider the simple case

$$f(t)=\exp(-t/\tau), \tag{34}$$

where τ is the switching time. Equation (33) is the dynamical equilibrium of the torques, when the inertial properties of the liquid crystal can be neglected [17], and in the following will be rewritten as

$$\frac{\partial^2\phi_E}{\partial z^2}-\frac{1}{\lambda^2}f(t)\phi_E=\frac{\eta}{k}\frac{\partial\phi_E}{\partial t}. \tag{35}$$

Equation (35) has to be solved by taking into account the boundary conditions $\phi_E(\pm d/2,t)=\phi_s$ and $\phi_E(z,0)=\Phi(z)$, where $\Phi(z)$ is solution of Eq. (3). We observe that from Eq. (35) it follows that

$$\left(\frac{\partial\phi_E}{\partial t}\right)_{t=0}=\frac{k}{\eta}\left(\frac{\partial^2\phi_E}{\partial z^2}-\frac{1}{\lambda^2}f(t)\phi_E\right)_{t=0}=\frac{k}{\eta}\left(\frac{d^2\Phi}{dz^2}-\frac{1}{\lambda^2}\Phi\right)=0, \tag{36}$$

i.e., the first-order time derivative of the tilt angle is automatically continuous at $t=0$. From Eq. (35) we obtain for the initial angular acceleration

$$\left(\frac{\partial^2\phi_E}{\partial t^2}\right)_{t=0}=-\frac{k}{\eta\lambda^2}\left(\frac{df}{dt}\right)_{t=0}\Phi(z). \tag{37}$$

From Eq. (37) it follows that the initial angular acceleration is positive. In the simple case where $f(t)$ is given by Eq. (34) we get

$$\left(\frac{\partial^2\phi_E}{\partial t^2}\right)_{t=0}=\frac{k}{\eta\lambda^2\tau}\Phi(z)=\frac{1}{\tau_\lambda\tau}\Phi(z), \tag{38}$$

from which it follows that the effective characteristic time is the geometrical average of τ_λ and τ.

Let us consider now the general solution of Eq. (35) with the relevant boundary conditions. We look for a solution of the type

$$\phi_E(z,t)=\phi_s+\sum_{n=0}^{\infty}K_n(t)\cos(a_n z), \tag{39}$$

where $K_n(0)=D_n$. By substituting Eq. (39) into Eq. (35), after simple calculations we obtain

$$\frac{dK_n}{dt}+H_n(t)K_n=-g_n(t), \tag{40}$$

where

$$H_n(t)=\frac{k}{\eta}\left(a_n^2+\frac{1}{\lambda^2}f(t)\right),$$

$$g_n(t)=4\frac{(-1)^n}{a_n d}\frac{k}{\eta\lambda^2}f(t)\phi_s. \tag{41}$$

The solution we are looking for $K_n(t)$ is

$$K_n(t)=e^{-u_n(t)}\left\{D_n-\int_0^t e^{u_n(t')}g_n(t')dt'\right\}, \tag{42}$$

where

$$u_n(t)=\int_0^t H_n(t')dt'. \tag{43}$$

In the case where $f(t)$ is given by Eq. (34) a simple calculation gives

$$u_n(t)=\frac{k}{\eta}\left\{\frac{\tau}{\lambda^2}(1-e^{-t/\tau})+a_n^2 t\right\}. \tag{44}$$

This relation shows that in the limit of large t, $u_n(t)\to(k/\eta)a_n^2 t$. This means that, for large t the relaxation times are the ones related to the diffusion times. In the opposite limit, where $t\to 0$, from Eq. (44) we get

$$u_n(t)=\frac{k}{\eta}\left\{\frac{1}{\lambda^2}+(2n+1)^2\left(\frac{\pi}{d}\right)^2\right\}t, \tag{45}$$

that for $n=0$ reads

$$u_0(t)=\frac{k}{\eta}\left\{\frac{1}{\lambda^2}+\left(\frac{\pi}{d}\right)^2\right\}t. \tag{46}$$

From this relation it follows that, for $\lambda\ll d$, the relaxation time is comparable with the diffusion time associated to the electric coherence length, τ_λ.

The solution of the problem, for generic t, can be easily obtained numerically by means of the formulas presented above. In the next section we will investigate the role of the switching time on the relaxation of the initial deformation.

VI. COMPARISON OF THE PREDICTIONS OF THE CONSIDERED MODELS

To compare the predictions of the model considered above we assume: $k=10^{-11}$ N, $\eta=10^{-1}$ Pa s, $d=10^{-6}$ m [12]. With these values $\tau_D=10^{-2}$ s. We suppose, furthermore, that the coherence length, related to the amplitude of the distorting field, is $\lambda=0.1d$. For the inertial properties we take 25×10^{-12} kg/m$\leq I\leq 25\times10^{-8}$ kg/m, which is very large with respect to real values for typical nematic liquid crystal [17], just to show the possible role of the inertial properties on the relaxation phenomenon. The numerical calculations have been performed by summation of the series of

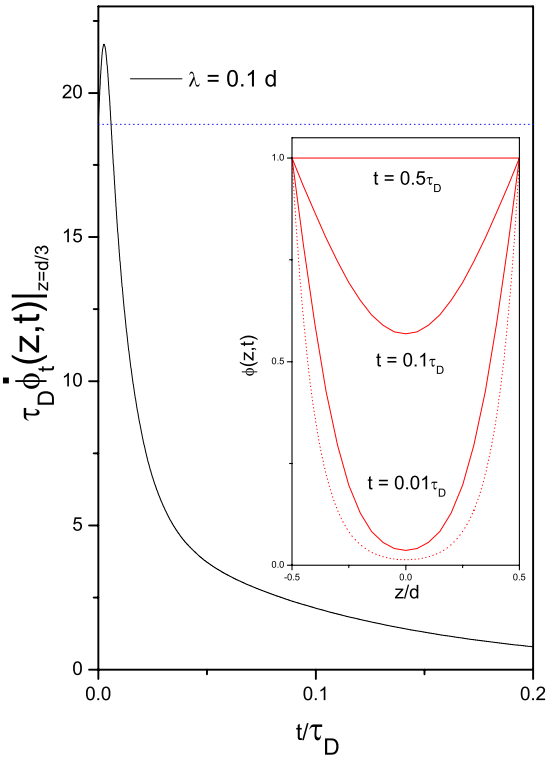

FIG. 1. (Color online) $\tau_D\dot{\phi}_t$ versus t/τ_D (where $\dot{\phi}_t \equiv \partial\phi/\partial t$) evaluated for $z=d/3$ for the diffusion model. The horizontal dotted blue line corresponds to this quantity evaluated by means of Eq. (17). The inset illustrates the behavior of the tilt profile angle for different times. The dotted red line is the initial profile of the system, $\Phi(z)$.

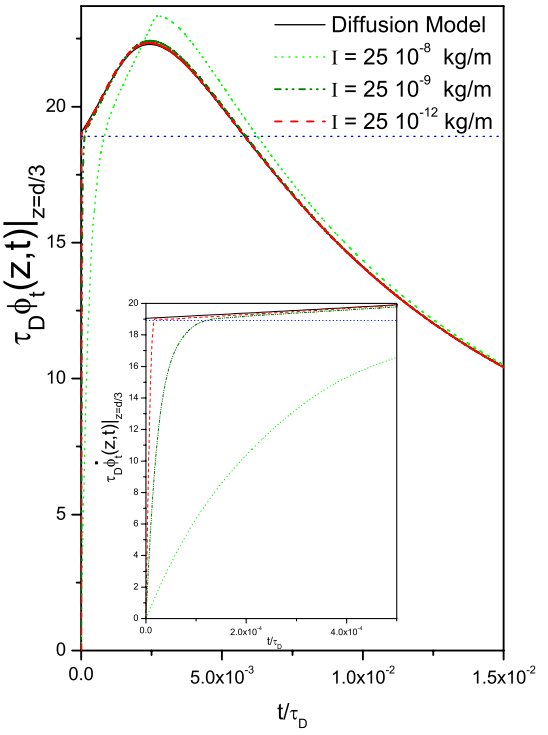

FIG. 2. (Color online) $\tau_D\dot{\phi}_t$ versus t evaluated for $z=d/3$ for the inertial model for two different values of I, in kg/m (dashed red and dotted green lines). The diffusion model (continuous black line) is also shown in this figure as well as the quantity evaluated by means of Eq. (17) (horizontal blue dotted line).

Fourier, with MATHEMATICA. The convergence of the series was tested by considering a large number of terms (100–500 terms). The number of terms used to perform the calculations was determined by means of the condition $(|\mathcal{S}_{n+1}-\mathcal{S}_n|<\epsilon \sim 10^{-8}$ where $S_n=\Sigma_{k=0}^{n}\mathcal{B}_k$, is the sum of the first n terms of the series, whose elements are indicated by $\mathcal{B}_k$.

In Fig. 1, we show the first-order time derivative of the tilt angle, at $z=d/3$, versus t derived with the model based on the equation of diffusion. In the same figure, the dotted line blue shows the value of this quantity evaluated by means of Eq. (17). The inset in Fig. 1 reports the tilt angle profile for different times, derived by means of the model of diffusion, expressed in terms of the diffusion time τ_D. The initial profile $\Phi(z)$ is represented by a dotted line red.

In Fig. 2, we show $\tau_D\dot{\phi}_t$ versus t (where $\dot{\phi}_t \equiv \partial\phi/\partial t$) evaluated for $z=d/3$ derived by means of the model where the inertial properties of the nematic molecules are taken into account, for $I=25\times10^{-8}$ kg/m, $I=25\times10^{-9}$ kg/m, and $I=25\times10^{-12}$ kg/m. Now the initial condition $\partial\phi/\partial t=0$ for $t=0$ is satisfied for all values of I, in contrast to the diffusion model. However, for small t the $\partial\phi/\partial t$ changes rapidly, as it follows from Eq. (30). For $t/\tau_D \geq 10^{-2}$ the predictions of the model taking into account the inertial properties of the molecules coincides, practically, with the model based on the diffusion equation. In the inset we show $\tau_D\dot{\phi}_t$ versus t for $t/\tau_D \leq 4\times10^{-4}$ for the values of I reported above. As expected, as $I\to0$, $\tau_D\dot{\phi}_t$ versus t presents a discontinuity for $t=0$.

In Fig. 3, we show for different times the profile tilt angle obtained with Eq. (23) with $I=25\times10^{-8}$ kg/m. The diffusion model, where the inertial properties of the liquid crystal have been neglected, is also illustrated in this figure. From this figure it follows that the tilt angle profiles predicted by the two models practically coincides for all t/τ_D.

In Fig. 4, we show $\tau_D\dot{\phi}_t$ versus t, derived with the model where the switching time of the distorting field is taken into account, evaluated for $z=d/3$. Now the initial condition $\partial\phi/\partial t=0$ for $t=0$ is automatically satisfied. This figure shows that for $\tau \ll \tau_D$ the predictions of the model taking into account the finite switching time of the distorting field tends to those of the model based on the diffusion equation.

In Fig. 5, we illustrate the profile for the tilt angle obtained in presence of an time dependent electric field and the diffusion model. In Fig. 6, we illustrate the prediction for $\tau_D\dot{\phi}_t$ versus t which is obtained with models analyzed above, evaluated for $z=d/3$.

In our paper we have considered the situation in which the distorting field is removed, and the initial nematic distortion $\phi(z,0)=\Phi(z)$ relaxes toward the undistorted configuration $\lim_{t\to\infty}\phi(z,t)=\phi_s$. Of course, a similar analysis can be done when the distorting field is switched on. In this case the initial boundary condition is $\phi(z,0)=\phi_s$, and for $t\to\infty$, $\lim_{t\to\infty}\phi(z,t)=\Phi(z)$. A simple extension of the presented investigation allows to investigate the evolution of the nematic

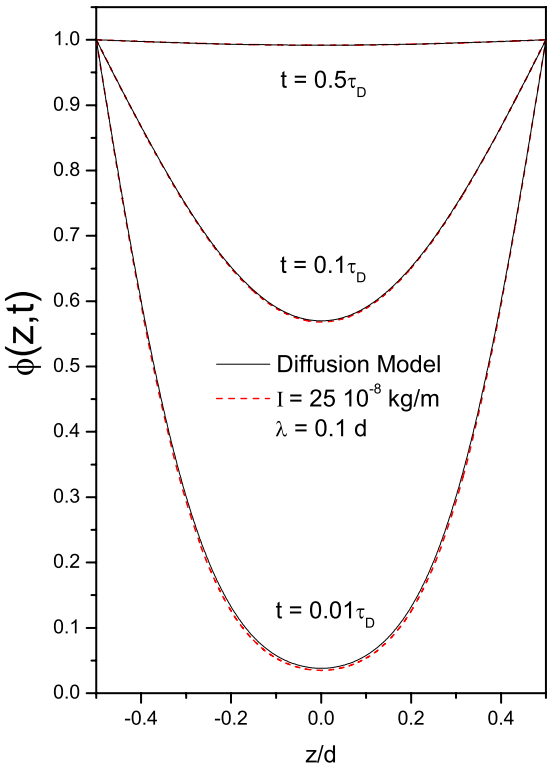

FIG. 3. (Color online) $\phi(z,t)$ versus z/d for the inertial (dashed red line) and diffusion (continuous black line) models for different values of t.

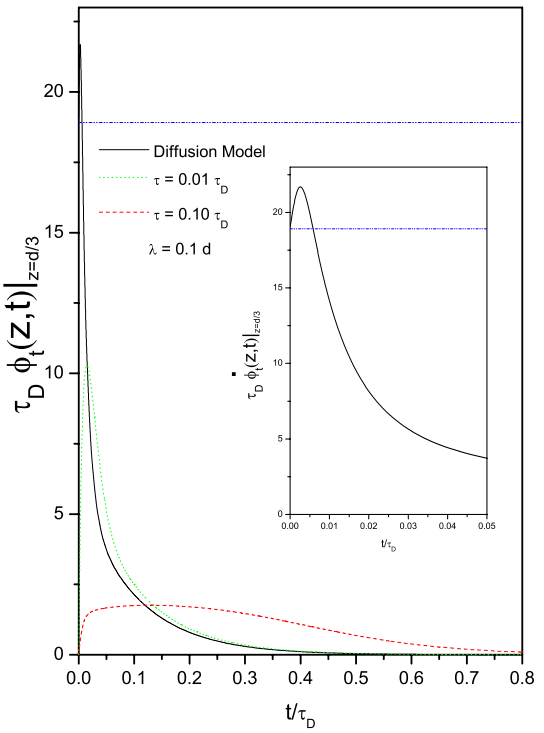

FIG. 4. (Color online) $\tau_D\dot{\phi}_t$ versus t/τ_D evaluated for $z=d/3$ for the case characterized by a distorting field for different relaxation times (dashed red and dotted green lines). The diffusion model (continuous black line) is also shown in this figure as well as the quantity evaluated by means of Eq. (17) (horizontal dashed-dotted blue line).

orientation when the easy direction is modified by means of an external action [18–24].

VII. CONCLUSIONS

We have investigated the relaxation of the nematic deformation when the distorting field is switched off. We have shown that the usual analysis based on the diffusion like equation does not allow a complete description of the phenomenon because it does not permit to satisfy the initial boundary condition, at $t=0$, on the first time derivative of the nematic tilt angle. According to this approach, the first time derivative of the nematic distortion is discontinuous at $t=0$, and the second-order time derivative presents, at $t=0$, has a δ-Dirac behavior. In this model, our analysis shows that for large t, the relaxation times are multiple of the diffusion time. We have also analyzed the model where the inertial properties of the nematic molecules are taken into account, in such a manner to satisfy the initial boundary conditions on the first-order time derivative of the tilt angle. In this framework the dynamical evolution of the nematic deformation, in the initial time, depends on the inertial properties of the nematic molecules, and the evolution toward the equilibrium state is still governed by the diffusion time. As discussed in the paper, both these approaches are based on the hypothesis that the distorting field is removed in a discontinuous manner at $t=0$, i.e., that the switching time is zero. We have shown that in a real case, where the switching time is of the order of the diffusion time, a rigorous analysis has to be done, taking into account the time dependence of the distorting field. In this framework, the continuity of the first time derivative of the nematic tilt angle is automatically satisfied, and for large t, the approaching to the equilibrium state is described by a multirelaxation phenomenon, whose characteristic times are proportional to the diffusion time. The evolution of the system from $t=0$ to the equilibrium state can differ strongly from the one predicted by the two models based on the diffusion equation, or taking into account the inertial properties of the nematic molecules.

APPENDIX

Let us consider a nematic liquid crystal submitted to an electric field. The electrostatic energy density due to the interaction between the electric field and the nematic material is given by [25] $f_e=(1/2)\mathbf{D}\cdot\mathbf{E}$, where the electric displacement $\mathbf{D}$ and the electric field $\mathbf{E}$ are related by the constitutive relation $\mathbf{D}=\epsilon\mathbf{E}$, where ϵ is the dielectric tensor. By assuming that the liquid crystal behaves as a perfect insulator, we have furthermore, in the quasistatic case, the equations of Maxwell $\nabla\cdot\mathbf{D}=0$, and $\nabla\times\mathbf{E}=0$. In the slab geometry considered in our analysis, from the Maxwell equations it follows that the electric field has only the z component different from zero, $\mathbf{E}=E\mathbf{z}$, where $\mathbf{z}$ is a unit vector along the z axis, and that the z component of the electric displacement, D, is constant. Consequently, from the constitutive equation we obtain

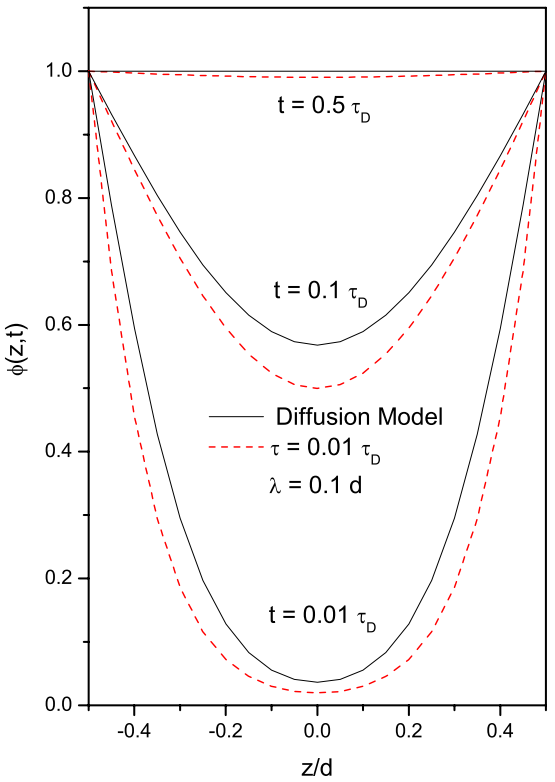

FIG. 5. (Color online) $\phi(z,t)$ versus z/d is illustrated for the case of a distorting field depending on t (dashed red line) and for the diffusion model (continuous black line), for different values of t.

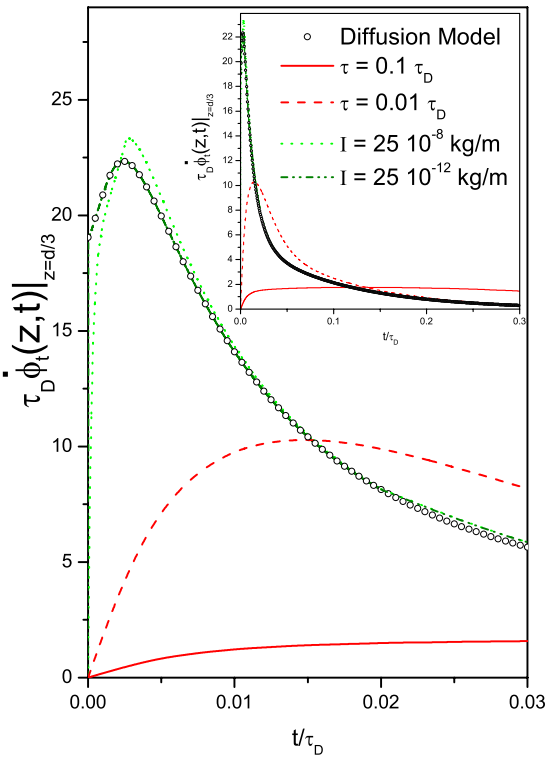

FIG. 6. (Color online) $\tau_D\dot{\phi}_t$ versus t/τ_D evaluated for $z=d/3$ for the three models worked out in the paper.

$D=\epsilon_{zz}E$, where $\epsilon_{zz}=\varepsilon_{\parallel}\cos^2\phi+\varepsilon_{\perp}\sin^2\phi$ is the effective dielectric constant along the z axis, and $\varepsilon_{\parallel}$ and $\varepsilon_{\perp}$ the dielectric constant along and perpendicular to the director, respectively. It follows that

$$E=\frac{D}{\varepsilon_{\parallel}-\varepsilon_a\sin^2\phi}, \tag{A1}$$

where $\varepsilon_a=\varepsilon_{\parallel}-\varepsilon_{\perp}$ is the dielectric anisotropy. Equation (A1) shows that the electric field is not constant across the sample as soon as $\varepsilon_a\neq 0$ [26]. In this framework, the electrostatic energy taking into account the interaction between the nematic liquid crystal and the external field is

$$f_e=\frac{1}{2}DE=\frac{1}{2}\epsilon_{zz}E^2=\frac{D^2}{2(\varepsilon_{\parallel}-\varepsilon_a\sin^2\phi)}. \tag{A2}$$

In the limit of small deformation considered in our paper, $\sin\phi\sim\phi$, and from Eq. (A2) we obtain, at the second order in ϕ,

$$f_e=\frac{D^2}{2\varepsilon_{\parallel}}+\frac{1}{2}\varepsilon_a\left(\frac{D}{\varepsilon_{\parallel}}\right)^2\sin^2\phi. \tag{A3}$$

The first addendum on the rhs is inessential in our problem because it is independent of the nematic orientation, and will be neglected in the following. Furthermore, the second term is quadratic in the ϕ, as the elastic energy density $f_d=(1/2)k(\partial\phi/\partial z)^2$. Consequently, it is enough to estimate D at the zeroth order in ϕ to have the total energy density $f=f_d+f_e$. To evaluate D, we use Eq. (A1). By integrating it over the thickness of the sample, and taking into account the definition of electric potential, we get

$$V=\int_{-d/2}^{d/2}Edz=\int_{-d/2}^{d/2}\frac{D}{\varepsilon_{\parallel}-\varepsilon_a\sin^2\phi}=\frac{D}{\varepsilon_{\parallel}}d\left(1+\frac{\varepsilon_a}{\varepsilon_{\parallel}}\langle\sin^2\phi\rangle\right), \tag{A4}$$

where V is the difference of potential between the electrode at $z=-d/2$ and that at $z=d/2$, and

$$\langle\sin^2\phi\rangle=\frac{1}{d}\int_{-d/2}^{d/2}\sin^2\phi dz. \tag{A5}$$

From Eq. (A4) we get, at the zeroth order in the nematic orientation, $D=\varepsilon_{\parallel}(V/d)$. This relation shows that, in the limit of small deformations, the z component of the electric field can be considered constant, and equal to $E_0=V/d$, and the electric energy density responsible for the electric torque acting on the nematic director is

$$f_e=\frac{1}{2}\varepsilon_aE_0^2\sin^2\phi, \tag{A6}$$

which is the expression used to derive the bulk differential equation in the Sec. II.

[1] I. Dozov, M. Nobili, and G. Durand, Appl. Phys. Lett. **70**, 1179 (1997).
[2] A. I. Derzhanski and A. G. Petrov, Acta Phys. Pol. A **55**, 747 (1979).
[3] M. Kleman and S. A. Pikin, J. Mech. **18**, 661 (1979).
[4] A. D. Rey, Macromolecules **24**, 4450 (1991).
[5] G. E. Durand and E. G. Virga, Phys. Rev. E **59**, 4137 (1999).
[6] A. M. Sonnet and E. G. Virga, Phys. Rev. E **61**, 5401 (2000).
[7] A. M. Sonnet, E. G. Virga, and G. E. Durand, Phys. Rev. E **62**, 3694 (2000).
[8] A. Mertelj and M. Copic, Phys. Rev. Lett. **81**, 5844 (1998).
[9] G. Barbero, I. Dhal, and L. Komitov, J. Chem. Phys. **130**, 174902 (2009).
[10] F. M. Leslie, Arch. Ration. Mech. Anal. **28**, 265 (1968).
[11] F. M. Leslie, Continuum Mech. Thermodyn. **4**, 5844 (1998).
[12] P. G. de Gennes, *The Physics of Liquid Crystals* (Oxford University Press, Oxford, 1974).
[13] A. A. Sonin, *The Surface Physics of Liquid Crystals* (Gordon and Breach, Philadelphia, 1995).
[14] G. Barbero and L. R. Evangelista, *An Elementary Course on the Continuum Theory for Nematic Liquid Crystals* (World Scientific, Singapore, 2001).
[15] D. W. Berreman, J. Appl. Phys. **46**, 3746 (1975).
[16] L. D. Landau and E. I. Lifchitz, *Theorie de l'elasticite* (MIR, Moscow, 1980).
[17] H. Kelker and R. Hatz, *Handbook of Liquid Crystals* (Verlag Chemie, Weinheim, 1980).
[18] P. Pieranski and P. Jerome, Mol. Cryst. Liq. Cryst. **179**, 285 (1990).
[19] O. V. Kuksenok and S. Shianovskii, Mol. Cryst. Liq. Cryst. Sci. Technol., Sect. A **359**, 107 (2001).
[20] I. Jánossy, Phys. Rev. E **49**, 2957 (1994).
[21] O. Francescangeli, S. Slussarenko, F. Simoni, D. Andrienko, V. Reshetnyak, and Yu. Reznikov, Phys. Rev. Lett. **82**, 1855 (1999).
[22] E. Ouskova, D. Fedorenko, Yu. Reznikov, S. V. Shiyanovskii, L. Su, J. L. West, O. V. Kuksenok, O. Francescangeli, and F. Simoni, Phys. Rev. E **63**, 021701 (2001).
[23] E. Ouskova, Yu. Reznikov, S. V. Shiyanovskii, L. Su, J. L. West, O. V. Kuksenok, O. Francescangeli, and F. Simoni, Phys. Rev. E **64**, 051709 (2001).
[24] D. Fedorenko, K. Slyusarenko, E. Ouskova, V. Reshetnyak, K. R. Ha, R. Karapinar, and Yu. Reznikov, Phys. Rev. E **77**, 061705 (2008).
[25] L. D. Landau and E. I. Lifchitz, *Electrodynamique des Milieux Continus* (MIR, Moscow, 1980).
[26] I. W. Stewart, *The Static and Dynamic Continuum Theory of Liquid Crystals: A Mathematical Introduction* (Taylor & Francis, London, 2004), Chap. 3, p. 89.